Lecture 18

Bruhat order / length corr

$W = \langle s_\alpha \; \alpha\in\Delta \mid \cdots \rangle \qquad \ell(x) = \min\{p \mid x = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_p}\}$

A word $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{\ell(x)}}$ representing x is said to be a minimal expression of x.

$x \le y$: there exists a minimal expr for y that has a subword that is a min expr for x

subword means: university $\longrightarrow$ university $\rightarrow$ net

Thm $x \le y \iff$ every min expr for y contains a min expr for x.

BB Cor 2.2.3

Now, there is a criterion for $x \le y$ in S_n that has a similar flavor to the positive root flip set I introduced earlier.

But it involves comparison of $\sim n^2$ integers, not a comparison of subsets of a set with $\sim \frac{n^2}{2}$ elements.

See BB Thm 2.1.5

Also, for S_n, $\ell(x) = |\Phi_x^+|$. (Prop 1.5.2).

Bruhat order for $\mathfrak{sl}_n\mathbb{C}$ = type A_{n-1}

$W = \mathrm{Sym}_n$ Simple roots $e_i - e_{i+1} \leftrightarrow$ perm $s_i \in W$ that exchanges i and $i+1$.

$BxB = ?$ Key obs: left action of B does upward row ops.

Hence if F^x = flag corresp to matrix x (or any in xB)

and E = std flag $e_1 \quad e_1e_2 \quad \cdots \quad e_1\ldots e_{n-1}$

Then mult on left by B doesn't change $d_{ij} = \dim(E_i \cap F_j^x)$

d_{ij} = # 1 entries in the upper-left $i \times j$ submatrix of w.

[Note: $x \le y \Leftrightarrow d^x_{ij} \ge d^y_{ij} \ \forall i,j$]

D = matrix w/ entries d_{ij} is "typically"

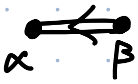
$\max(i+j-n, 0) \leftarrow$ w_0 case, $w_0 = (n \ n-1 \ n-2 \ldots$

but could be as large as $\min(i,j) \leftarrow$ e case!

So $C_w \subset \{F \mid \dim E_i \cap F_j = d_{ij}\}$ hence $X_w \subset \{F \mid \dim E_i \cap F_j \ge d_{ij}\}$

in fact $=$ in fact $=$

Bruhat order for B_2 / $sp(4,\mathbb{C})$

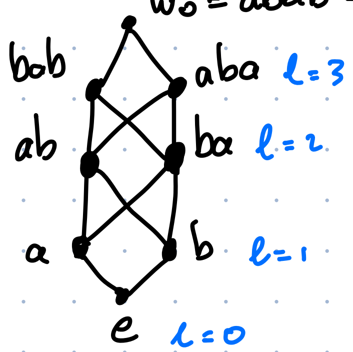


$W \cong \underset{\uparrow \text{ swap}}{S_2} \ltimes \underbrace{(\mathbb{Z}/2)^2}_{\text{binary vector}}$

$a = s_\alpha$ corresp to the S_2 generator

$b = s_\beta$ corresp to the flip of one elt e.g. $(0,1)$

$w_0 = abab = baba$ $\ell = 4$



G/B = isotropic flags in $(\mathbb{R}^4, \omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix})$

$= \{(p, \ell),\ p$ in $\mathbb{C}P^3$, ℓ line in $\mathbb{R}P^3$ that is isotropic $\}$

(p_0, ℓ_0)

X_{aba}, X_{bab} ? $\{(p,\ell) \mid p \in p_0^\perp\}$ $\{(p,\ell) \mid \ell$ meets $\ell_0\}$

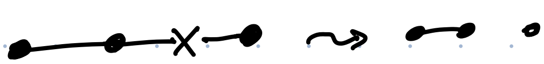
X_a, X_b ? $\{(p,\ell) \mid \ell = \ell_0\}$ and ... ?

Parabolic subgroups of W

$\Theta \subset \Delta$ W_Θ = subgrp of W gen by s_α, $\alpha \in \Theta$

recall P_Θ = std parabolic whose negative roots are $\mathbb{Z}$-span $\Theta \cap \Phi^-$

W_Θ is also a Coxeter group!



Thm (Cor 2.4.5 in BB) Each coset xW_{Θ} is convex an has rep of min length and of max length.

There is an induced order on the min length reps

$\to$ this is the diag of Schubert cells in G/P.

$$G/P = \bigcup_{x \in W/W_P} BxP$$

Ex. $\mathbb{CP}^2$ as quot of $Flag(\mathbb{C}^3)$.
